CHAPTER 9

LINES

A straight line is the next simplest geometric object after a point. All physical examples of a straight line are finite line segments with well-defined endpoints and length. However, the mathematical or geometric line may be unbounded or infinite, or it may be a semi-infinite half-line or ray. This chapter reviews the mathematical description of lines in two and three dimensions, including linear parametric equations. It describes how to compute points on a line, geometric relationships between points and lines, line intersections, and translating and rotating lines.

9.1 Lines in the Plane

The *slope-intercept form* is algebraically the simplest way to describe a straight line that lies in the x, y plane (Figure 9.1). If we know the slope m of a line and where it intersects the y axis, at $y = b$, then we write the equation of the line as

$$y = mx + b \tag{9.1}$$

The slope m is the ratio of the change in y to the change in x between any two points on the line. If the coordinates of these two points are x_1, y_1 and x_2, y_2, then the slope of the line that passes through them is

$$m = \frac{x_2 - x_1}{y_2 - y_1} \tag{9.2}$$

The *point-slope form* is a variation of the slope intercept form (Figure 9.2). If we know the slope m and a point x_1, y_1 through which the line passes, then

$$y - y_1 = m(x - x_1) \tag{9.3}$$

If we use the point $x_1 = 0$, $y_1 = b$, then Equation 9.3 simplifies to Equation 9.1.

The *two-point form* derives from the proposition that any two distinct points in the plane define a line that passes through both of them (Figure 9.3). Thus, given two

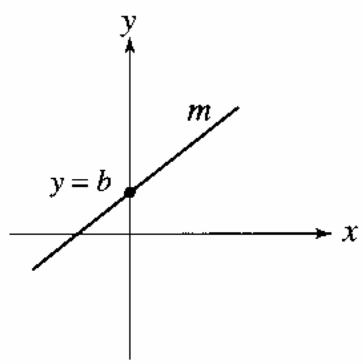

Figure 9.1 *Slope-intercept form.*

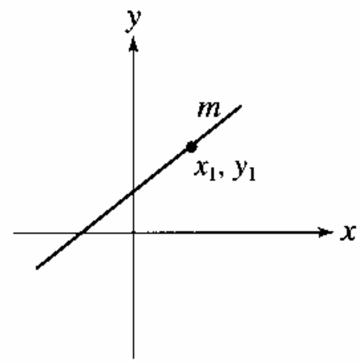

Figure 9.2 *Point-slope form.*

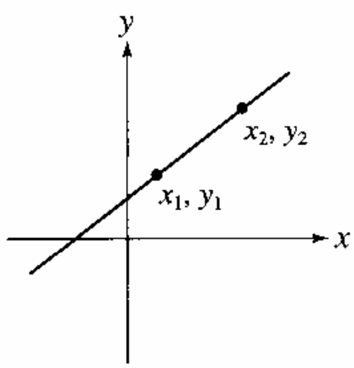

Figure 9.3 *Two-point form.*

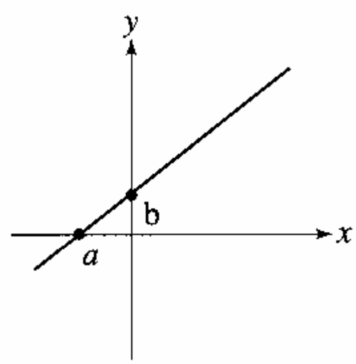

Figure 9.4 *Intercept form.*

points x_1, y_1 and x_2, y_2, we substitute Equation 9.2 into Equation 9.3 for m, and rearrange the terms to obtain

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \tag{9.4}$$

Using Equation 9.4 for the two points $(x_1, y_1) = (a, 0)$ and $(x_2, y_2) = (0, b)$, produces

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{9.5}$$

This is the *intercept form*, aptly named, since it defines a line by its points of intersection with the coordinate axes (Figure 9.4).

Equations 9.1 and 9.3 are *explicit* equations, where x is the *independent variable* and y is the *dependent variable*. That is, we choose arbitrary values for x and compute the corresponding values of y. The value of y depends on the value of x. Equations 9.4 and 9.5 are *implicit equations*, where we can choose either x or y to be the independent variable. However, we can express x and y separately in terms of a third variable, say u. Using this approach we need two equations to define a line in the plane. They are

$$\begin{aligned} x &= au + b \\ y &= cu + d \end{aligned} \tag{9.6}$$

These are *parametric equations*, and u is the *parametric variable* . Ordinarily, we treat u as the independent variable and x and y as dependent variables.

The parametric equations of a line in the plane work like this: Let's say we want to define a line that passes through x_1, y_1 and x_2, y_2 and that $u = 0$ at the first point, $u = 1$ at the second point. This is enough information to write two sets of two simultaneous equations to determine the constant coefficients $a, b, c,$ and d. In this case, for $u = 0$ in Equations 9.6, we have

$$\begin{aligned} x_1 &= b \\ y_1 &= d \end{aligned} \quad \text{or} \quad \begin{aligned} b &= x_1 \\ d &= y_1 \end{aligned} \tag{9.7}$$

and for $u = 1$

$$\begin{aligned} a + b &= x_2 \\ c + d &= y_2 \end{aligned} \tag{9.8}$$

We find b and d directly, from Equations 9.7, and substitute appropriately into Equations 9.8 to find a and c. Thus

$$\begin{aligned} x_2 &= a + x_1 \\ y_2 &= c + y_1 \end{aligned} \quad \text{or} \quad \begin{aligned} a &= x_2 - x_1 \\ c &= y_2 - y_1 \end{aligned} \tag{9.9}$$

Now we know $a, b, c,$ and d for the line passing through the two given points, and we rewrite Equations 9.6 to obtain the parametric equations of this line:

$$\begin{aligned} x &= (x_2 - x_1)u + x_1 \\ y &= (y_2 - y_1)u + y_1 \end{aligned} \tag{9.10}$$

To find a set of points on this line, we substitute a set of u values into Equations 9.10. Each u value determines a coordinate pair of x, y values. This way of defining straight lines is often used in computer graphics and geometric modeling, because it allows a sequence of points on the line to be computed and plotted on a display screen. These are *linear parametric equations* . We use higher-order parametric equations to define curves and surfaces. Of course, the extension to three dimensions is obvious.

Because parametric equations allow us to separate the dependent x and y coordinate values of points along a line, the method readily lends itself to vector geometry (see Chapter 1).

We can express the angle θ between two lines in the plane as a function of their two respective slopes, m_1 and m_2 (Figure 9.5):

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{9.11}$$

The derivation of Equation 9.11 is a simple exercise in algebra and trigonometry.

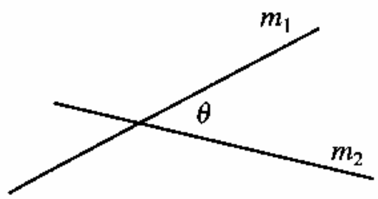

Figure 9.5 *Angle between two lines.*

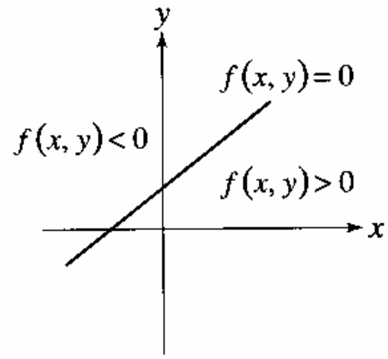

Figure 9.6 *Halfspaces defined by the implicit equation of a line.*

If two lines are parallel, then

$$m_1 = m_2 \tag{9.12}$$

If they are perpendicular, then

$$m_1 m_2 = -1 \tag{9.13}$$

A *halfspace* is defined by the implicit equation of a line (Figure 9.6). For example, given the explicit equation of the line $y = x + 2$, the corresponding implicit equation is $f(x, y) = x - y + 2$. If for any point x, y, $f(x, y) = 0$, then the point lies on the line. If $f(x, y) > 0$ or $f(x, y) < 0$, the point lies on one side or the other of the line. We say that $f(x, y) > 0$ and $f(x, y) < 0$ define two halfspaces, and $f(x, y) = 0$ defines their common boundary (Chapter 7).

9.2 Lines in Space

Three linear parametric equations, one for each coordinate, define a straight line in three-dimensional space:

$$\begin{aligned} x &= a_x u + b_x \\ y &= a_y u + b_y \\ z &= a_z u + b_z \end{aligned} \tag{9.14}$$

where x, y, and z are the dependent variables. Equations 9.14 generate a set of coordinates for each value of the parametric variable u. The coefficients a_x, a_y, a_z, b_x, b_y, and b_z are unique and constant for any given line, their values depending on the endpoint coordinates.

We can think of this set of equations as a point-generating machine. The input is values of u. The machine produces coordinates of points on a line as output (Figure 9.7). It produces a bounded line segment if we limit the range of values we assign to the parametric variable. In computer graphics and geometric modeling, u usually takes

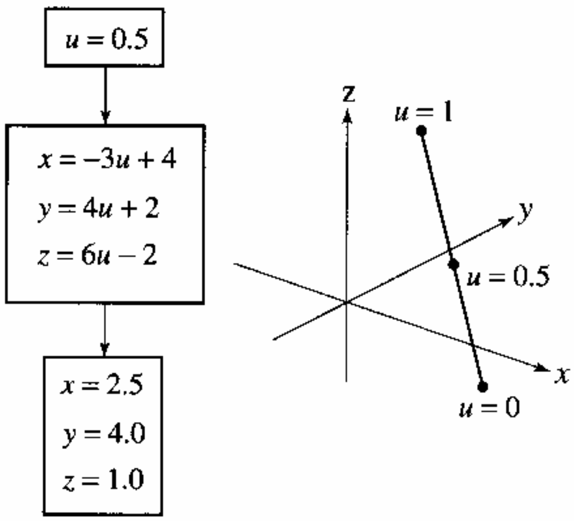

Figure 9.7 *A point-generating machine.*

values in the closed interval from 0 to 1. This is called the *unit interval*, and is expressed as

$$u \in [0, 1] \tag{9.15}$$

The interval limits determine the nature of the line, making it a line segment, a semi-infinite line (a ray), or an infinite line. If we insert values of $u = 0$ and $u = 1$ into the point-generating machine of Figure 9.7, we obtain the endpoint coordinates $\mathbf{p}_0 = (4, 2, -2)$ and $\mathbf{p}_1 = (1, 6, 4)$. To characterize a line segment by the coordinates of its endpoints, we must modify Equations 9.14, identifying the endpoints of the line segment, as above, by $\mathbf{p}_0$ and $\mathbf{p}_1$ (Figure 9.8).

Substituting $u = 0$ into Equations 9.14 yields

$$\begin{aligned} b_x &= x_0 \\ b_y &= y_0 \\ b_z &= z_0 \end{aligned} \tag{9.16}$$

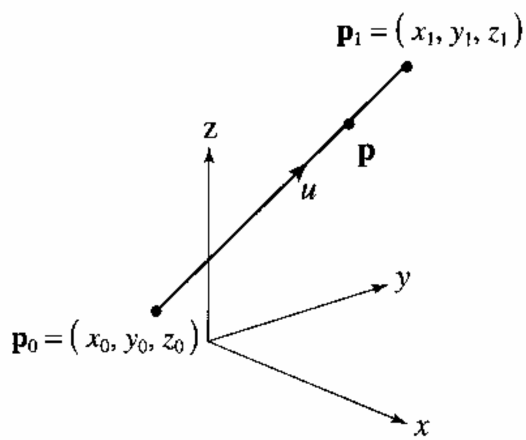

Figure 9.8 *Line segment in space.*

where x_0, y_0, and z_0 are the coordinates at the $u = 0$ endpoint of the line. At $u = 1$,

$$\begin{aligned} x_1 &= a_x + x_0 \\ y_1 &= a_y + y_0 \\ z_1 &= a_z + z_0 \end{aligned} \tag{9.17}$$

or

$$\begin{aligned} a_x &= x_1 - x_0 \\ a_y &= y_1 - y_0 \\ a_z &= z_1 - z_0 \end{aligned} \tag{9.18}$$

Substituting the results of Equations 9.16 and 9.18 into Equations 9.14 yields

$$\begin{aligned} x &= (x_1 - x_0)u + x_0 \\ y &= (y_1 - y_0)u + y_0 \qquad u \in [0, 1] \\ z &= (z_1 - z_0)u + z_0 \end{aligned} \tag{9.19}$$

This is a very useful set of equations. It means that if we know the endpoint coordinates of a line segment, we can immediately write a parametric equation for it and find any intermediate points on it. The parametric variable conveniently ranges through the closed interval from zero to one.

To find the length of a line segment we simply apply the Pythagorean theorem to the endpoint coordinate differences:

$$L = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} \tag{9.20}$$

There are three numbers associated with every line that uniquely describe its angular orientation in space. These numbers are called *direction cosines* (or *direction numbers*). We denote them as d_x, d_y, and d_z. The computations are simple:

$$\begin{aligned} d_x &= \frac{(x_1 - x_0)}{L} \\ d_y &= \frac{(y_1 - y_0)}{L} \\ d_z &= \frac{(z_1 - z_0)}{L} \end{aligned} \tag{9.21}$$

The geometry of the direction cosine computation for d_x is shown in Figure 9.9, and also for d_y and d_z. The edges of the auxiliary rectangular solid are parallel to the coordinate axes, and $\mathbf{p}_0$, A, and $\mathbf{p}_1$ define a right angle at A. From this figure we see that

$$d_x = \cos\theta \tag{9.22}$$

The sum of the squares of the direction cosines must equal one. Therefore, any two of them are sufficient to determine the third:

$$d_x^2 + d_y^2 + d_z^2 = 1 \tag{9.23}$$

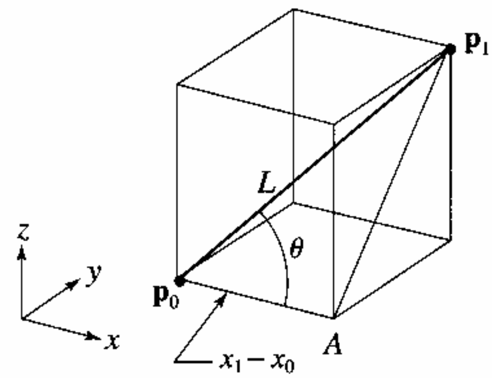

Figure 9.9 *Geometry for the computation of a direction cosine.*

However, direction cosines alone do not completely specify a line, because they tell us nothing about the location of the line. For example, any two parallel lines in space have the same direction cosines. But direction cosines are a good way to test whether they are parallel.

9.3 Computing Points on a Line

The parametric equations of a line are particularly useful for computing points on that line. An important set of points on a line frequently used in geometric modeling and computer graphics is points at equal intervals along the line. There are two ways to compute the coordinates for these points, one considerably faster and more efficient than the other. We will look at both methods.

To find the coordinates of points at n equal intervals on a given line, we use $\mathbf{p}_0$ and $\mathbf{p}_1$ and then compute the remaining $n-1$ intermediate points. Figure 9.10 shows a line with eight equal intervals.

The first method uses Equation 9.19, for each of the $n-1$ points. There are $n-1$ values of the parametric variable, given by

$$u = \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-2}{n}, \frac{n-1}{n} \tag{9.24}$$

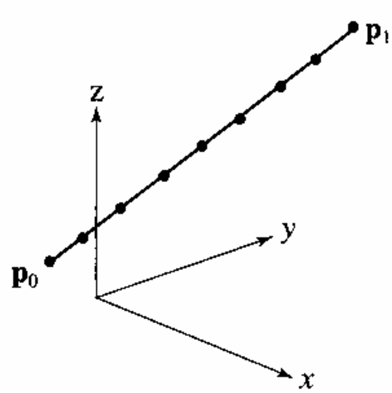

Figure 9.10 *Points at equal intervals along a line segment.*

and requiring $n - 1$ divisions to compute. For the x coordinates there are $n - 1$ multiplications and n additions (including finding $x_1 - x_0$). The computation totals for all coordinates are $3n$ additions, $3(n - 1)$ multiplications, and $n - 1$ divisions.

A second approach uses fewer computations. First, we notice that u always changes by a constant amount Δu, where

$$\Delta u = \frac{1}{n} \tag{9.25}$$

Next, we observe that for any x_i

$$x_i = (x_1 - x_0)u_i + x_0 \tag{9.26}$$

and for x_{i+1}

$$x_{i+1} = (x_1 - x_0)u_{i+1} + x_0 \tag{9.27}$$

However, since $u_{i+1} = u_i + \Delta u$, we can rewrite Equation 9.27 as

$$x_{i+1} = (x_1 - x_0)(u_i + \Delta u) + x_0 \tag{9.28}$$

or

$$x_{i+1} = (x_1 - x_0)u_i + (x_1 - x_0)\Delta u + x_0 \tag{9.29}$$

But $(x_1 - x_0)u_i + x_0 = x_i$, so we can simplify Equation 9.29:

$$x_{i+1} = x_i + (x_1 - x_0)\Delta u \tag{9.30}$$

and since $(x_1 - x_0)\Delta u$ is a constant, we let $\Delta x = (x_1 - x_0)\Delta u$. Therefore,

$$x_{i+1} = x_i + \Delta x \tag{9.31}$$

This tells us that we find each successive x coordinate by adding a constant to the previous value. This derivation applies to the y and z coordinates as well. Now let's count the computations: To compute Δu requires one division. To compute Δx requires one addition and one multiplication. For the x coordinates there are $n - 1$ additions. The totals for all n points are $3n$ additions, 3 multiplications, and one division. This process is called the *forward difference* method. It is commonly used to compute points on curves and surfaces, too.

9.4 Point and Line Relationships

Any point $\mathbf{q}$ is either on or off a given line (Figure 9.11). If it is on the line, it is either between the endpoints, $\mathbf{q}_1$, on the backward extension of the line, $\mathbf{q}_2$, or on the forward extension, $\mathbf{q}_3$.

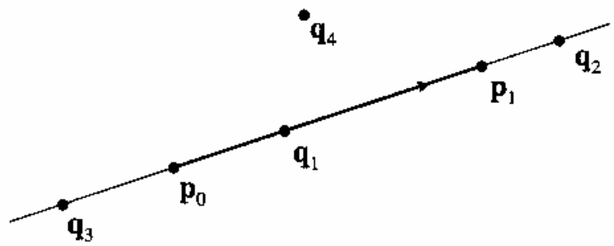

Figure 9.11 *Point and line relationships.*

We can write Equations 9.19 in terms of u to obtain

$$u_x = \frac{x - x_0}{x_1 - x_0}$$
$$u_y = \frac{y - y_0}{y_1 - y_0} \tag{9.32}$$
$$u_z = \frac{z - z_0}{z_1 - z_0}$$

Given the coordinates of any point $\mathbf{q} = (x, y, z)$, we compute u_x, u_y, and u_z. If and only if $u_x = u_y = u_z$ is point $\mathbf{q}$ on the line. Otherwise, it is off the line. We should allow some small but finite deviation. For example, $|u_x - u_y| = \varepsilon$, where $\varepsilon \ll 1$. If point $\mathbf{q}$ is on the line, then the value of u indicates its precise position.

In a plane, we can determine the position of a point relative to a line by solving the parametric equations to obtain the implicit equation

$$f(x, y) = (x - x_0)(y_1 - y_0) - (y - y_0)(x_1 - x_0) \tag{9.33}$$

Then, for a reference point $\mathbf{p}_R$ not on the line, we compute $f(x_R, y_R)$. For an arbitrary test point $\mathbf{p}_T$ we compute $f(x_T, y_T)$. If $f(x_T, y_T) = 0$, then $\mathbf{p}_T$ is on the line. If $f(x_R, y_R)$ and $f(x_T, y_T)$ have the same sign, that is, $f(x_T, y_T) > 0$ and $f(x_R, y_R) > 0$, or $f(x_T, y_T) < 0$ and $f(x_R, y_R) < 0$, then $\mathbf{p}_T$ and $\mathbf{p}_R$ are on the same side of the line. Otherwise they are on opposite sides (Figure 9.12). One way to choose a reference point is to let $\mathbf{p}_R = (x_0 + 1, y_0)$; then $\mathbf{p}_R$ is to the right of the line. If $y_0 = y_1$, the line is horizontal, and we can then let $\mathbf{p}_R = (x_0, y_0 + 1)$, placing $\mathbf{p}_R$ above the line.

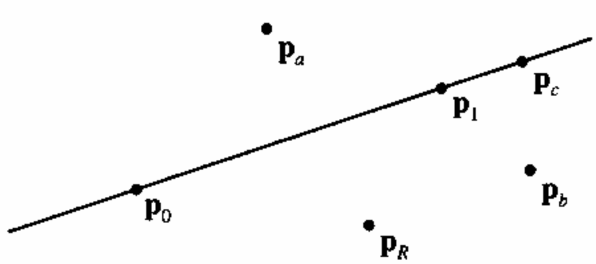

Figure 9.12 *Position of a point relative to a line.*

9.5 Intersection of Lines

If two lines intersect, they have a common point, the *point of intersection*. There are two problems of interest: first, the general problem of determining if two lines in space intersect (Figure 9.13a); second, the special case of finding the intersection of a line in the plane with a second line in the plane that is either horizontal or vertical (Figure 9.13b).

For the general problem, two lines, a and b, intersect if there is a point (the point of intersection) such that

$$\begin{aligned} x_a &= x_b \\ y_a &= y_b \\ z_a &= z_b \end{aligned} \tag{9.34}$$

It follows from Equations 9.19 that

$$\begin{aligned} (x_{1a} - x_{0a})u_a + x_{0a} &= (x_{1b} - x_{0b})u_b + x_{0b} \\ (y_{1a} - y_{0a})u_a + y_{0a} &= (y_{1b} - y_{0b})u_b + y_{0b} \\ (z_{1a} - z_{0a})u_a + z_{0a} &= (z_{1b} - z_{0b})u_b + z_{0b} \end{aligned} \tag{9.35}$$

where x_{0a} and y_{0a} are the coordinates of the endpoint $u = 0$ of line a, x_{1a} and y_{1a} are the coordinates of the endpoint $u = 1$ of line a, and similarly for line b.

We can use any two of the three equations to solve for u_a and u_b. Then we substitute u_a and u_b into the remaining equation to verify the solution. If the solution is verified, the lines intersect. Finally, both u_a and u_b must be in the interval 0 to 1 for a valid intersection.

For the special problem, given a line in the x, y plane, determine if it intersects with either of the vertical lines W_R, W_L (right or left), or with either of the horizontal lines W_T, W_B (top or bottom) (Figure 9.14). (Note that this nomenclature anticipates the window boundary coordinates of computer graphic displays: Chapter 14.) W_R is a

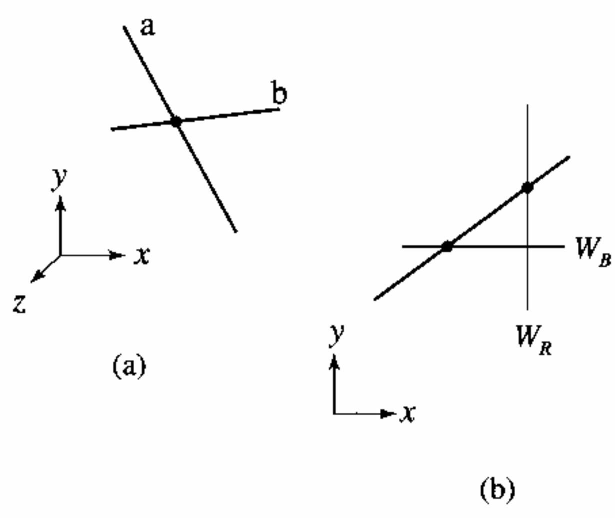

Figure 9.13 *Intersection of lines.*

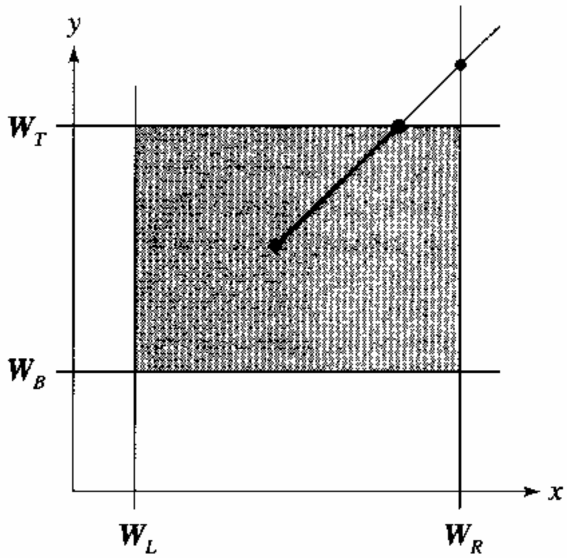

Figure 9.14 *Intersections with horizontal and vertical lines.*

vertical line whose equation is $x = W_R$. W_T is a horizontal line whose equation is $y = W_T$. If $x = W_R$ intersects the arbitrary line, then

$$u = \frac{W_R - x_0}{x_1 - x_0} \tag{9.36}$$

If Equation 9.36 produces a value of u in the unit interval, we use it to compute the y coordinate of the point of intersection. If the arbitrary line itself is vertical, then it cannot intersect the line $x = W_R$ (for a vertical line $x_0 = x_1$ and, therefore, $x_1 - x_0 = 0$). We use a similar procedure when computing line intersections with W_L, W_T, and W_B.

9.6 Translating and Rotating Lines

We can translate a line by translating its end-point coordinates (Figure 9.15). The end-point translations must be identical, otherwise the transformed line will have a different length or angular orientation, or both.

In the plane, we have

$$\begin{aligned} x_0' &= x_0 + x_T \qquad & x_1' &= x_1 + x_T \\ y_0' &= y_0 + y_T \qquad & y_1' &= y_1 + t_T \end{aligned} \tag{9.37}$$

The translated line is always parallel to its original position, and its length does not change. In fact, all points on the line are translated equally. That is why this is called a *rigid body translation*. The generalization to three or more dimensions is straightforward.

The simplest rotation of a line in two dimensions is about the origin (Figure 9.16). To do this we rotate both endpoints $\mathbf{p}_0$ and $\mathbf{p}_1$ through an angle θ about the origin. Then we find the coordinates of the transformed endpoints $\mathbf{p}_0'$ and $\mathbf{p}_1'$ by

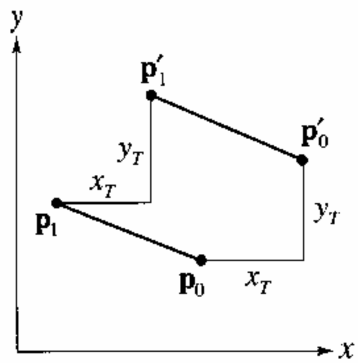

Figure 9.15 *Translating a line.*

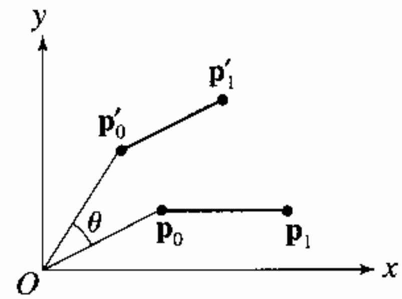

Figure 9.16 *Rotation of a line about a point in the plane.*

applying Equation 8.13:

$$\begin{aligned} x'_0 &= x_0 \cos\theta - y_0 \sin\theta \qquad & x'_1 &= x_1 \cos\theta - y_1 \sin\theta \\ y'_0 &= x_0 \sin\theta + y_0 \cos\theta \qquad & y'_1 &= x_1 \sin\theta + y_1 \cos\theta \end{aligned} \tag{9.38}$$

This equation produces a rigid-body rotation of the line. Again, a rotation in three dimensions is a direct extension of this (see Section 3.5).

Exercises

9.1 Prove that the sum of the squares of the direction cosines of a line is equal to one.

9.2 Compute the length and direction cosines for each of the following line segments, defined by their endpoints:
a. $\mathbf{p}_0 = (3.7, 9.1, 0.2)$, $\mathbf{p}_1 = (0.9, -2.6, 2.6)$
b. $\mathbf{p}_0 = (2.1, -6.4, 0)$, $\mathbf{p}_1 = (3.3, 0.7, -5.1)$
c. $\mathbf{p}_0 = (10.3, 4.2, 3.7)$, $\mathbf{p}_1 = (6.0, 10.3, 9.2)$
d. $\mathbf{p}_0 = (5.3, -7.9, 1.4)$, $\mathbf{p}_1 = (0, 4.1, 0.7)$

9.3 Write the parametric equations for each line given in Exercise 9.2.

9.4 How does reversing the order of the end points defining a line segment affect its length and direction cosines?

9.5 Compute the $n-1$ intermediate points on each of the following line segments, defined by their endpoints:
a. $\mathbf{p}_0 = (7, 3, 9)$, $\mathbf{p}_1 = (7, 3, 0)$, for $n = 3$
b. $\mathbf{p}_0 = (-4, 6, 0)$, $\mathbf{p}_1 = (2, 11, -7)$, for $n = 4$
c. $\mathbf{p}_0 = (0, 0, 6)$, $\mathbf{p}_1 = (6, 1, -5)$, for $n = 4$

9.6 Given the line segment defined by its endpoints $\mathbf{p}_0 = (6, 4, 8)$ and $\mathbf{p}_1 = (8, 8, 12)$, write the parametric equations of this line segment and determine for each of the following points if it is on or off the line segment.

a. $\mathbf{q}_1 = (4, 0, 8)$
b. $\mathbf{q}_2 = (12, -8, 20)$
c. $\mathbf{q}_3 = (7, 6, 10)$
d. $\mathbf{q}_4 = (10, 4, 2)$
e. $\mathbf{q}_5 = (10, 12, 16)$

9.7 Write the parametric equations for each of the line segments whose endpoints are given, and find the point of intersection (if any) between each pair of line segments.

a. Line 1: $\mathbf{p}_0 = (2, 4, 6)$, $\mathbf{p}_1 = (4, 6, -4)$
Line 2: $\mathbf{p}_0 = (0, 0, 1)$, $\mathbf{p}_1 = (6, 8, -6)$
b. Line 1: $\mathbf{p}_0 = (2, 4, 6)$, $\mathbf{p}_1 = (4, 6, -4)$
Line 2: $\mathbf{p}_0 = (4, 3, 5)$, $\mathbf{p}_1 = (2.5, 4.5, 3.5)$
c. Line 1: $\mathbf{p}_0 = (2, 4, 6)$, $\mathbf{p}_1 = (4, 6, -4)$
Line 2: $\mathbf{p}_0 = (3, 5, 1)$, $\mathbf{p}_1 = (0, 2, 16)$
d. Line 1: $\mathbf{p}_0 = (10, 8, 0)$, $\mathbf{p}_1 = (-1, -1, 0)$
Line 2: $\mathbf{p}_0 = (13, 2, 0)$, $\mathbf{p}_1 = (4, 7, 0)$
e. Line 1: $\mathbf{p}_0 = (5, 0, 0)$, $\mathbf{p}_1 = (2, 0, 0)$
Line 2: $\mathbf{p}_0 = (8, 0, 0)$, $\mathbf{p}_1 = (10, 0, 0)$